

Duality in Supersymmetric $SU(N_c)$ Gauge Theory with Two Adjoint Chiral Superfields

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We discuss $SU(N_c)$ gauge theory coupled to two adjoint chiral superfields X and Y , and a number of fundamental chiral superfields Q^i . We add a superpotential that has the form of Arnold's D series $W = \text{Tr } X^{k+1} + \text{Tr } XY^2$. We present a dual description in terms of an $SU(3kN_f - N_c)$ gauge theory, and we show that the duality passes many tests. At the end of the paper, we show how a deformation of this superpotential flows to another duality having a product gauge group $SU(N_c) \times SU(N'_c)$, with an adjoint field charged under $SU(N_c)$, an adjoint field charged under $SU(N'_c)$, fields in the (N_c, N'_c) and $(\overline{N}_c, \overline{N}'_c)$ representation, and a number of fundamentals. The dual description is an $SU(2kN'_f + kN_f - N'_c) \times SU(2kN_f + kN'_f - N_c)$ gauge theory.

1. Introduction.

Drawing on the work of [1-9], N. Seiberg was able to show that two different four dimensional $N = 1$ supersymmetric gauge theories can flow to the same non-trivial fixed point or that a theory can flow to a dual free magnetic phase [10]. Since then, many more examples of such dualities have been discovered with matter in the fundamental representation of the gauge group [11-12]. A few duals have been found for chiral theories involving more matter than just fundamentals [13-15]. In general, theories with two index tensors have proved difficult to solve. For example, a general dual theory was searched for in the case of $SU(N_c)$ gauge theory with N_f fundamentals and one adjoint, X , but to no avail. Instead, a special version of the theory was solved by Kutasov et al. [16-18] by adding a superpotential of the form

$$W = \text{Tr } X^{k+1}. \quad (1.1)$$

This superpotential does three things. It reduces the number of gauge invariant operators in the theory by setting $X^k = 0$, it fixes the R-charge of the field X , and, because X^{k+1} is a dangerously irrelevant operator, it presumably takes the theory from the poorly understood fixed point, to a new fixed point that can be solved. Since the solution of this one adjoint model, other examples of dualities involving one two-index tensor were found [19-24]. In this paper, like Kutasov et al., we will employ a superpotential, but here we have fundamentals and *two* adjoint fields. The superpotential we will add looks like

$$W = \text{Tr } X^{k+1} + \text{Tr } XY^2. \quad (1.2)$$

The dual gauge group is $SU(3kN_f - N_c)$. It is interesting to note that the form of the superpotential looks like the D_{k+2} classification of Arnold's singularity theory. However, in our case the X and Y field carry gauge indices and therefore do not commute. Nevertheless, it is tempting to speculate that perhaps there is a relation between singularity theory and conformal field theories in four dimensions as there is in two dimensions. Such a connection between Arnold's A_k classification and the superpotential (1.1) was exploited in [18].

At the end of this paper, we will show how the superpotential can be deformed and flow to a prediction for another duality with a product gauge group $SU(N_c) \times SU(N'_c)$ and a superpotential

$$W = \text{Tr } X_1^{k+1} + \text{Tr } X_2^{k+1} + \text{Tr } X_1 F \tilde{F} + \text{Tr } X_2 F \tilde{F}. \quad (1.3)$$

where adjoint field X_1 is charged under $SU(N_c)$, adjoint field X_2 is charged under $SU(N'_c)$, and fields F and \tilde{F} are in the (N_c, N'_c) representation. There are also N_f fundamentals and N_f anti-fundamentals Q and \tilde{Q} that are charged under only $SU(N_c)$ and N_f fundamentals and N_f anti-fundamentals Q' and \tilde{Q}' that are charged under only $SU(N'_c)$. The gauge group that we propose as dual to this theory is $SU(2kN'_f + kN_f - N'_c) \times SU(2kN_f + kN'_f - N_c)$

2. The Theory

The theory we will discuss in this paper will be a supersymmetric non-Abelian gauge theory with gauge group $SU(N_c)$, two chiral superfields X and Y which transform under the adjoint representation of the gauge group, N_f fundamentals Q^i , N_f anti-fundamentals \tilde{Q}^i where $i = 1, \dots, N_f$. The theory is asymptotically free for $N_c > N_f$. We will also add a superpotential of the form

$$W = \frac{s_1}{k+1} \text{Tr } X^{k+1} + s_2 \text{Tr } XY^2 + \lambda_1 \text{Tr } X + \lambda_2 \text{Tr } Y \quad (2.1)$$

where λ_1 and λ_2 are Lagrange multipliers to implement the tracelessness condition on the adjoint matter fields X and Y . This superpotential (2.1) is not corrected by non-perturbative effects for all values of N_f [5]. The equations of motion that follow from this superpotential are

$$\begin{aligned} s_1 X^k + s_2 Y^2 + \lambda_1 &= 0 \\ s_2 (XY + YX) + \lambda_2 &= 0. \end{aligned} \quad (2.2)$$

These equations truncate the chiral ring for odd and values of k . To illustrate this we can ignore λ_1 and λ_2 ; statements below will then be modulo lower order terms already included in the chiral ring. We can multiply X^k by Y from the right or left, and use the first equation in (2.2) to show that

$$YX^k + X^kY = -2 \frac{s_2}{s_1} Y^3. \quad (2.3)$$

Now, we can use the second equation in (2.2) to anticommute the Y fields through the X fields.

$$((-1)^k + 1)X^k Y = -2 \frac{s_2}{s_1} Y^3 \quad (2.4)$$

Thus for odd k , $Y^3 = 0$. The chiral ring is now said to be truncated because we can relate higher order operators to lower ones. From $Y^3 = 0$ and equation (2.2), it follows that in theories with superpotentials having odd values of k for $k < N_c$, all gauge invariants can be formed from products of the Q , the \tilde{Q} , and $X^{l-1}Y^{j-1}$ where $l = 1, 2, \dots, k$; $j = 1, 2, 3$. We can make mesons of the form

$$(M_{lj})^i_i = \tilde{Q}_i X^{l-1} Y^{j-1} Q^i; \quad l = 1, 2, \dots, k; \quad j = 1, 2, 3 \quad (2.5)$$

We can also form baryons by introducing dressed quarks

$$Q_{(l,j)} = X^{l-1} Y^{j-1} Q; \quad l = 1, \dots, k; \quad j = 1, 2, 3. \quad (2.6)$$

and then contracting the gauge indices on an epsilon tensor.

$$B^{(n_{1,1}, n_{2,1}, \dots, n_{k,3})} = Q_{(1,1)}^{n_{1,1}} \cdots Q_{(k,3)}^{n_{k,3}}; \quad \sum_{l=1}^k \sum_{j=1}^3 n_{l,j} = N_c \quad (2.7)$$

The total number of baryons is

$$\sum_{\{n_{l,j}\}} \binom{N_f}{n_{1,1}} \cdots \binom{N_f}{n_{k,3}} = \binom{3kN_f}{N_c}. \quad (2.8)$$

We can also make invariants of the form $\text{Tr } X^{l-1} Y^{j-1}$. When $j = 2$, or $j = 3$ and l is even, the trace operators can be set to zero by using the equations of motion and the cyclic property of traces.

3. Duality

The theory that we are proposing as dual to the one described above has an $SU(3kN_f - N_c)$ gauge group, two chiral superfields \bar{X} and \bar{Y} which transform under the adjoint representation of the gauge group, N_f fundamentals q^i , N_f anti-fundamentals \tilde{q}^i where $i = 1, \dots, N_f$, and M_{lj} singlets which are in a one-to-one mapping with the mesons of equation (2.5). The dual superpotential has the form

$$W = s_1 \text{Tr } \bar{X}^{k+1} + s_2 \text{Tr } \bar{X} \bar{Y}^2 + \frac{s_1 s_2}{\mu^4} \sum_{l=1}^k \sum_{j=1}^3 M_{lj} \tilde{q} \bar{X}^{k-l} \bar{Y}^{3-j} q. \quad (3.1)$$

Both theories have the following anomaly free global symmetries:

$$SU(N_f) \times SU(N_f) \times U(1)_B \times U(1)_R \quad (3.2)$$

The chiral superfields of the electric theory transform as follows under these global symmetries:

$$\begin{aligned} Q & (N_f, 1, 1, 1 - \frac{N_c}{N_f(k+1)}) \\ \tilde{Q} & (1, \bar{N}_f, -1, 1 - \frac{N_c}{N_f(k+1)}) \\ X & (1, 1, 0, \frac{2}{k+1}) \\ Y & (1, 1, 0, \frac{k}{k+1}). \end{aligned} \quad (3.3)$$

The dual matter fields transform as follows under the global symmetries (3.2):

$$\begin{aligned} q & (\bar{N}_f, 1, \frac{N_c}{3kN_f - N_c}, 1 - \frac{3kN_f - N_c}{N_f(k+1)}) \\ \tilde{q} & (1, N_f, -\frac{N_c}{3kN_f - N_c}, 1 - \frac{3kN_f - N_c}{N_f(k+1)}) \\ \bar{X} & (1, 1, 0, \frac{2}{k+1}) \\ \bar{Y} & (1, 1, 0, \frac{k}{k+1}) \\ M_{lj} & (N_f, \bar{N}_f, 0, 2 - \frac{2}{k+1} \frac{N_c}{N_f} + \frac{2(l-1) + k(j-1)}{k+1}) \end{aligned} \quad (3.4)$$

Again $l \leq k$ and $j \leq 3$. This dual theory is asymptotically free for $3kN_f - N_c > N_f$ otherwise it is in a free magnetic phase. The superpotential (3.1) is invariant under the flavor symmetries and baryon number. This is a non-trivial check of the mapping of the mesons (2.5) into the singlet fields M_{lj} . For the superpotential be invariant under the R-symmetry, it was essential to have the chiral ring truncated as described above and have the mapping of mesons to elementary singlet fields as described in equation (2.5). Baryon-like objects are mapped to other baryon-like objects in the dual theory. The mapping is

$$\begin{aligned} B_{\text{el}}^{(n_{1,1}, n_{2,1}, \dots, n_{k,3})} & \leftrightarrow B_{\text{mag}}^{(m_{1,1}, m_{2,1}, \dots, m_{k,3})}; \quad m_{l,j} = N_f - n_{k+1-l,4-j}; \\ & l = 1, 2, \dots, k; \quad j = 1, 2, 3 \end{aligned} \quad (3.5)$$

The fact that this mapping is consistent with all global symmetries is another non-trivial test of the proposed duality. Traces of products of X and Y are mapped to the same traces of products of \bar{X} and \bar{Y} .

The t'Hooft anomaly matching conditions are satisfied for arbitrary k . They are

$$\begin{aligned}
SU(N_f)^3 & N_c d^{(3)}(N_f) \\
SU(N_f)^2 U(1)_R & -\frac{1}{k+1} \frac{N_c^2}{N_f} d^{(2)}(N_f) \\
SU(N_f)^2 U(1)_B & N_c d^{(2)}(N_f) \\
U(1)_R & -\frac{1}{k+1} (N_c^2 + 1) \\
U(1)_R^3 & \left(\left(\frac{2}{k+1} - 1 \right)^3 + \left(\frac{k}{k+1} - 1 \right)^3 + 1 \right) (N_c^2 - 1) - \frac{2}{(k+1)^3} \frac{N_c^4}{N_f^2} \\
U(1)_B^2 U(1)_R & -\frac{2}{k+1} N_c^2.
\end{aligned} \tag{3.6}$$

The t'Hooft anomaly matching conditions, especially the $U(1)_R^3$ condition, are very powerful tests of the duality.

By considering only the global symmetries, we can find a scale matching relation between the electric theory scale Λ and the magnetic theory scale $\bar{\Lambda}$ involving only the dimensionful parameter μ and the coupling constants s_1 and s_2 .

$$\Lambda^{N_c - N_f} \bar{\Lambda}^{\bar{N}_c - N_f} = C s_1^{-3N_f} s_2^{-3kN_f} \mu^{4N_f} \tag{3.7}$$

where C is some constant. Such scale matching relations are common in dual theories.

3.1. $k=1$.

Before we start deforming the superpotential, let's consider the $k=1$ case. From the superpotential,

$$W = \frac{s_1}{2} \text{Tr } X^2 + s_2 \text{Tr } XY^2 + \lambda_1 \text{Tr } X + \lambda_2 \text{Tr } Y, \tag{3.8}$$

we see that X is massive and below the scale s_1 , we can integrate it out. Using the equations of motion for X ,

$$s_1 X + s_2 Y^2 + \lambda_1, \tag{3.9}$$

we see that the superpotential becomes

$$W = \frac{3s_2^2}{2s_1} \text{Tr } Y^4 + \frac{2s_2}{s_1} \lambda_1 \text{Tr } Y^2 + \lambda_2 \text{Tr } Y. \tag{3.10}$$

We recall that λ_1 is a Lagrange multiplier which should be integrated out. By taking the trace of (3.9), we see that

$$\lambda_1 = s_2 \text{Tr } Y^2. \tag{3.11}$$

Inserting this into (3.10) , we find

$$W = \frac{3s_2^2}{2s_1} \text{Tr } Y^4 + \frac{2s_2^2}{s_1} (\text{Tr } Y^2)^2 + \lambda_2 \text{Tr } Y, \quad (3.12)$$

which is actually a marginal deformation of Kutasov's $k = 3$ duality (1.1). Mesons involving the X field are massive. Hence, we will be left with only $Q\tilde{Q}, QY\tilde{Q}, QY^2\tilde{Q}$, and a dual $SU(3N_f - N_c)$ gauge group which is correct for Kutasov's duality [17]. The scale matching relations (3.7) flows to the scale matching found in [18]

$$\Lambda^{2N_c - N_f} \bar{\Lambda}^{2N_c - N_f} = C \left(\frac{\mu_L}{s_0} \right)^{2N_f}, \quad (3.13)$$

where $s_0 = \frac{s_2^2}{s_1}$ from (3.12) and $\mu_L^2 = \frac{s_0 \mu^4}{s_1 s_2}$ from (3.1) and [18].

3.2. Theories with superpotentials having even k

As we saw in section 2, the theories with a superpotential with k odd have a chiral ring that is truncated by the classical equations of motion. This is not the case for theories with a superpotential having even values of k . Nevertheless, as we have seen in this section, all of the checks on the duality work for k even or odd. Thus, it is tempting to speculate that the duality also holds for theories with even values of k . The chiral ring for the k even theories might be truncated by some quantum mechanical mechanism. We will see in the next section that if we make the assumption that the duality hold for both even and odd values of k , we will naturally be led to another duality.

4. Deformations.

4.1. Giving a mass to M .

We will now check that the duality and the scale matching relation (3.7) are consistent with the renormalization group flow from a theory with N_f flavors to one with $N_f - 1$ flavors. Upon giving a mass to one of the quarks in the electric theory,

$$W = \frac{s_1}{k+1} \text{Tr } X^{k+1} + s_2 \text{Tr } XY^2 + m\tilde{Q}_{N_f} Q^{N_f}, \quad (4.1)$$

we can integrate out the massive meson and flow to a theory with $N_f - 1$ flavors. The relation between the high energy scale, Λ_{N_c, N_f} , and the low energy scale, $\Lambda_{N_c, N_f - 1}$ is

$$m\Lambda_{N_c, N_f}^{N_c - N_f} = \Lambda_{N_c, N_f - 1}^{N_c - N_f + 1}. \quad (4.2)$$

When we give a mass to $\tilde{Q}_{N_f} Q^{N_f}$, we must also add the corresponding singlet field on the magnetic side:

$$W = s_1 \text{Tr } \bar{X}^{k+1} + s_2 \text{Tr } \bar{X} \bar{Y}^2 + \frac{s_1 s_2}{\mu^4} \sum_{l=1}^k \sum_{j=1}^3 M_{lj} \tilde{q} \bar{X}^{k-l} \bar{Y}^{3-j} q + m(M_{1,1})_{N_f}^{N_f}. \quad (4.3)$$

Upon integrating out the singlet field M , we find that one of the mesons on the dual side has acquired a non-zero vacuum expectation value.

$$\tilde{q}^{N_f} \bar{X}^{k-1} \bar{Y}^2 q_{N_f} = -\frac{m\mu^4}{s_1 s_2} \quad (4.4)$$

For $k = 2$, a solution that satisfies the F-terms, D-terms, and equation (4.4), while giving no other meson a mass is

$$\bar{X} = \begin{pmatrix} 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & . \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & . \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & . \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & . \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & . \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & . \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & . \\ . & . & . & . & . & . & . & . \end{pmatrix}$$

$$\bar{Y} = \begin{pmatrix} 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & . \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & . \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & . \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & . \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & . \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & . \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & . \\ . & . & . & . & . & . & . & . \end{pmatrix}$$

with $\tilde{q}_\alpha^{N_f} = 2\delta_{\alpha,1}$ and $q_{N_f}^\alpha = 2\delta^{\alpha,6}$. For arbitrary k , the solution will take the form $\bar{X} = \bar{X}_{\beta-1}^\alpha$ with $\bar{X}_{k+1}^k = 0, \bar{X}_{2k+1}^{2k} = 0$, and all the other elements zero except for \bar{X}_{2k+1}^k ; $\bar{Y} = \bar{Y}_{\beta-k}^\alpha$ and $\tilde{q}_\alpha^{N_f} = \delta_{\alpha,1}$ and $q_{N_f}^\alpha = \delta^{\alpha,3k}$ with all other elements zero. Given this ansatz it is quite easy to insert it into (4.4), (2.2), and the D-flat equations to determine the values of non-zero matrix elements exactly up to gauge transformations. The exact solution for $k = 3$ is given in the appendix. The vevs for \bar{X} and \bar{Y} , Higgses the gauge group from $SU(\bar{N}_c)$ down to $SU(\bar{N}_c - 3k)$. The q_{N_f} and \tilde{q}_{N_f} fields that received vevs will also be eaten. The adjoint fields \bar{X} and \bar{Y} will break apart into the smaller adjoint fields of the $SU(\bar{N}_c - 3k)$ gauge group plus $6k$ fundamentals, $3k - 1$ of which will be eaten by the Higgs

mechanism and $3k + 1$ of which will receive a mass. Thus, the magnetic theory will flow at low energies to a theory that is dual to the low energy electric theory.

In order to simplify the scale matching calculations, it is convenient to make the assumption that the vacuum expectation value for X and Y are the same. This calculation can also be done when X and Y do not have the same vev. From equation (2.2), this assumption implies a relation between s_1 and s_2 :

$$s_1 = s_2 \left(\frac{m\mu^4}{s_1 s_2} \right)^{\frac{2-k}{k+3}}. \quad (4.5)$$

The scale matching works in three stages. At the first stage, the gauge vector bosons acquire a mass $(\frac{m\mu^4}{s_1 s_2})^{\frac{1}{k+3}}$. Thus, the scale matching for this stage is

$$\bar{\Lambda}_{\bar{N}_c, N_f}^{\bar{N}_c - N_f} = \left(\frac{m\mu^4}{s_1 s_2} \right)^{\frac{6k}{k+3}} \bar{\Lambda}_{\bar{N}_c - 3k, N_f + 3k}^{\bar{N}_c - 6k - N_f}. \quad (4.6)$$

In the second stage, the $3k + 1$ fundamentals coming from the decomposition of the adjoint matter fields receive a mass from the superpotential terms

$$W_{\text{mag}} = \frac{s_1}{k+1} \text{Tr } \bar{X}^{k+1} \simeq s_1 \langle \bar{X} \rangle^{k-1} \bar{X}^2 = s_1 \left(\frac{m\mu^4}{s_1 s_2} \right)^{\frac{k-1}{k+3}} \bar{X}^2 = s_2 \left(\frac{m\mu^4}{s_1 s_2} \right)^{\frac{1}{k+3}} \bar{X}^2, \quad (4.7)$$

$$W_{\text{mag}} = s_2 \text{Tr } \bar{X} \bar{Y}^2 \simeq s_2 \langle \bar{X} \rangle \bar{Y}^2 = s_2 \left(\frac{m\mu^4}{s_1 s_2} \right)^{\frac{1}{k+3}} \bar{Y}^2, \quad (4.8)$$

and

$$W_{\text{mag}} = s_2 \text{Tr } \bar{X} \bar{Y}^2 \simeq s_2 \langle \bar{Y} \rangle \bar{X} \bar{Y} = s_2 \left(\frac{m\mu^4}{s_1 s_2} \right)^{\frac{1}{k+3}} \bar{X} \bar{Y}, \quad (4.9)$$

where we have used (4.5) in (4.7). Flowing down in energy, the scale matching relation is

$$\bar{\Lambda}_{\bar{N}_c - 6k, N_f}^{\bar{N}_c - 6k - N_f} = s_2^{-3k-1} \left(\frac{m\mu^4}{s_1 s_2} \right)^{\frac{-3k-1}{k+3}} \bar{\Lambda}_{\bar{N}_c - 3k, N_f - 1}^{\bar{N}_c - 3k - N_f + 1} \quad (4.10)$$

Combining equation (4.6) and (4.10) we find

$$\bar{\Lambda}_{\bar{N}_c, N_f}^{\bar{N}_c - N_f} = s_2^{-3k-1} \left(\frac{m\mu^4}{s_1 s_2} \right)^{\frac{3k-1}{k+3}} \bar{\Lambda}_{\bar{N}_c - 3k, N_f - 1}^{\bar{N}_c - 3k - N_f + 1}. \quad (4.11)$$

Rewriting this as

$$\bar{\Lambda}_{\bar{N}_c, N_f}^{\bar{N}_c - N_f} = s_2^{-3k-1} \left(\frac{m\mu^4}{s_1 s_2} \right)^{\frac{2k-4}{k+3}} \left(\frac{m\mu^4}{s_1 s_2} \right) \bar{\Lambda}_{\bar{N}_c - 3k, N_f - 1}^{\bar{N}_c - 3k - N_f + 1}, \quad (4.12)$$

we see that we can use (4.5) to replace $\left(\frac{m\mu^4}{s_1 s_2}\right)^{\frac{2k-4}{k+3}}$ with $\left(\frac{s_2}{s_1}\right)^2$. Thus, we find that the scale matching relation of the low energy theory is

$$\bar{\Lambda}_{\bar{N}_c, N_f}^{\bar{N}_c - N_f} = s_1^{-3} s_2^{-3k} m \mu^4 \bar{\Lambda}_{\bar{N}_c - 3k, N_f - 1}^{\bar{N}_c - 3k - N_f + 1}. \quad (4.13)$$

Now, by using the scale matching relation between the electric and the magnetic theory for N_f flavors (3.7), the relation between the electric theory scale with N_f flavors and $N_f - 1$ flavors (4.2), and the relation between the magnetic theory scale with N_f flavors and $N_f - 1$ flavors (4.13), we can derive a relation between the electric theory scale and the magnetic theory scale with $N_f - 1$ flavors. It is

$$\Lambda^{N_c - N_f + 1} \bar{\Lambda}^{\bar{N}_c - N_f + 1} = C s_1^{-3N_f + 3} s_2^{-3k(N_f - 1)} \mu^{4N_f - 4} \quad (4.14)$$

where C is some constant. By comparing with (3.7), we see easily the scale matching relation has been preserved under the renormalization group flow.

The duality map between baryons (3.5) should more properly be accompanied by a scale matching relation that can be determined from the global symmetries and the flow described above.

$$B^{i_1 \dots i_{n_{1,1}}, j_1 \dots j_{n_{2,1}}, \dots, z_1 \dots z_{n_{k,3}}} = P \left(\prod_{i=1}^3 \prod_{j=1}^k \frac{1}{\bar{n}_{i,j}!} \right) s_1^{\frac{N_c + 3N_f k}{2}} s_2^{\frac{3k}{2}(N_c + N_f) - N_c} \Lambda^{\frac{3k}{2}(N_c - N_f)} \mu^{-2\bar{N}_c} \\ \epsilon^{i_1 \dots i_{n_{1,1}}, \bar{z}_1 \dots \bar{z}_{\bar{n}_{k,3}}} \epsilon^{j_1 \dots j_{n_{2,1}}, \bar{y}_1 \dots \bar{y}_{\bar{n}_{k-1,3}}} \dots \epsilon^{z_1 \dots z_{n_{k,3}}, \bar{i}_1 \dots \bar{i}_{\bar{n}_{1,1}}} \bar{B}_{\bar{i}_1 \dots \bar{i}_{\bar{n}_{1,1}}, \dots, \bar{y}_1 \dots \bar{y}_{\bar{n}_{k-1,3}}, \bar{z}_1 \dots \bar{z}_{\bar{n}_{k,3}}} \quad (4.15)$$

P is some phase. When a flavor on the electric side is given a mass, those baryons that contained that quark become heavy and are not present in the low energy theory. On the dual side the gauge group breaks, and to preserve the baryon mapping, we must saturate, the first $3k$ gauge indices, with the expectation values of the fields. Thus the mapping between the high energy and low energy baryons on the dual side becomes

$$\epsilon^{i_1 \dots i_{n_{1,1}}, \bar{z}_1 \dots \bar{z}_{\bar{n}_{k,3}}} \epsilon^{j_1 \dots j_{n_{2,1}}, \bar{y}_1 \dots \bar{y}_{\bar{n}_{k-1,3}}} \dots \epsilon^{z_1 \dots z_{n_{k,3}}, \bar{i}_1 \dots \bar{i}_{\bar{n}_{1,1}}} \bar{B}_{\bar{i}_1 \dots \bar{i}_{\bar{n}_{1,1}}, \dots, \bar{y}_1 \dots \bar{y}_{\bar{n}_{k-1,3}}, \bar{z}_1 \dots \bar{z}_{\bar{n}_{k,3}}}^{\bar{N}_c, N_f} \rightarrow \\ \bar{n}_{1,1} \bar{n}_{2,1} \dots \bar{n}_{k,3} \left(\frac{m\mu^4}{s_1 s_2} \right)^{\frac{3k}{2}} \epsilon^{i_1 \dots i_{n_{1,1}}, \bar{z}_1 \dots \bar{z}_{\bar{n}_{k,3}-1}} \dots \epsilon^{z_1 \dots z_{n_{k,3}}, \bar{i}_1 \dots \bar{i}_{\bar{n}_{1,1}-1}} \bar{B}_{\bar{i}_1 \dots \bar{i}_{\bar{n}_{1,1}-1}, \dots, \bar{z}_1 \dots \bar{z}_{\bar{n}_{k,3}-1}}^{\bar{N}_c - 6, N_f - 1} \quad (4.16)$$

Using equation (4.2) we see that these relations are just right to preserve the duality mapping between baryonic operators (4.15) under the renormalization group flow from the theory with N_f flavor to the theory with $N_f - 1$ flavors.

4.2. Flat directions of adjoint fields for odd values of k

It is convenient to consider the theories with even and odd values of k separately. We consider the k odd case first.

$$W = \frac{s_1}{k+1} \text{Tr } X^{k+1} + s_2 \text{Tr } XY^2 - \lambda_1 \text{Tr } X - \lambda_2 \text{Tr } Y. \quad (4.17)$$

The equations of motion

$$\begin{aligned} s_1 X^k + s_2 Y^2 - \lambda_1 &= 0 \\ XY + YX - \lambda_2 &= 0. \end{aligned} \quad (4.18)$$

An $SU(2n + km)$ gauge theory will have a flat direction along which X gets a vacuum expectation value

$$X = a \begin{pmatrix} 0_n & 0 & 0 & 0 & 0 & 0 & \cdot & 0 \\ 0 & 0_n & 0 & 0 & 0 & 0 & \cdot & 0 \\ 0 & 0 & 1_m & 0 & 0 & 0 & \cdot & 0 \\ 0 & 0 & 0 & \omega_m & 0 & 0 & \cdot & 0 \\ 0 & 0 & 0 & 0 & \omega_m^2 & 0 & \cdot & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega_m^3 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdot & \omega_m^{k-1} \end{pmatrix}$$

$$Y = b \begin{pmatrix} 1_n & 0 & 0 & 0 & 0 & 0 & \cdot & 0 \\ 0 & -1_n & 0 & 0 & 0 & 0 & \cdot & 0 \\ 0 & 0 & 0_m & 0 & 0 & 0 & \cdot & 0 \\ 0 & 0 & 0 & 0_m & 0 & 0 & \cdot & 0 \\ 0 & 0 & 0 & 0 & 0_m & 0 & \cdot & 0 \\ 0 & 0 & 0 & 0 & 0 & 0_m & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdot & 0_m \end{pmatrix},$$

where $\omega = \exp \frac{2\pi i}{k}$ and $a = \sqrt[k]{\frac{\lambda_1}{s_1}}$ and $b = \sqrt[2]{\frac{\lambda_2}{s_2}}$. The subscripts on the entries indicate that they are matrices. One could think of the Y field's vacuum expectation value breaking the theory into three, and then the X field vev breaking one of those three into k parts. We end up with $SU(n) \times SU(n) \times SU(m)^k \times U(1)^{k+1}$. The fields X and Y each decompose into $k+2$ adjoints and fields in the (n, n) , (n, m) , (m, n) , and (m, m) representations. We will call the fields, coming from X in the (n, n) representation, F . It can be shown F does not receive a mass from the superpotential and that there is the term

$$W_L = \frac{s_1}{k+1} \text{Tr } (F \widetilde{F})^{\frac{k+1}{2}} \quad (4.19)$$

All the other matter fields (except the Q s) will receive a mass upon decomposition of the superpotential. Thus along this flat direction the theory flows to a duality discussed in a paper by Intriligator, Leigh, and Strassler [24] in the $SU(n) \times SU(n)$ part and Seiberg's SQCD [10] in the $SU(m)^k$ part. On the dual side, the dual gauge group was $SU(3kN_f - N_c)$. It breaks to $SU(kN_f - n) \times SU(kN_f - n) \times SU(N_f - m)^k \times U(1)^{k+1}$ which is just right for preserving the duality.

It can be shown that the scale matching relation (3.7) and the scale matching relation for SQCD are consistent with this flat direction.

It was explained in [24] how to deform (4.19) by lower order operators and flow to SQCD. One can think of deformations of (4.19) in the high energy theory as adding even powered operators $\text{Tr } X^r$ to the superpotential (4.17).

4.3. Deformations of superpotential by adjoint fields for k even.

We now move on and look at the case where k is even. From section 2, we remember that this was the case in which the classical superpotential did not truncate the chiral ring sufficiently for the proposed duality to be valid. Assuming duality, additional constraints must be present. To implement these constraints we add to our superpotential some Lagrange multipliers, setting in particular the operators $\text{Tr } Y^j$ to zero where $j > 2$. The superpotential looks like

$$W = s_1 \text{Tr } X^{k+1} + \text{Tr } XY^2 - \lambda_1 \text{Tr } X - \lambda_2 \text{Tr } Y + \frac{\eta_3}{3} \text{Tr } Y^3 + \frac{\eta_4}{4} \text{Tr } Y^4 + \dots \quad (4.20)$$

where the η s are the Lagrange multipliers. Let's now deform the superpotential by X operators of odd powers only (that are in this modified chiral ring). The superpotential becomes

$$W = \sum_{r=1}^{\frac{k}{2}} \frac{g_r}{2r+1} \text{Tr } X^{2r+1} + \text{Tr } XY^2 - \lambda_1 \text{Tr } X - \lambda_2 \text{Tr } Y + \frac{\eta_3}{3} \text{Tr } Y^3 + \frac{\eta_4}{4} \text{Tr } Y^4 + \dots \quad (4.21)$$

The equations of motion from (4.21) are

$$\begin{aligned} \sum_{r=1}^{\frac{k}{2}} g_r X^{2r} + s_2 Y^2 - \lambda_1 &= 0 \\ XY + YX - \lambda_2 + \eta_3 Y^2 + \eta_4 Y^3 + \dots &= 0. \end{aligned} \quad (4.22)$$

Because we deformed the superpotential by lower order terms of odd powers only, we see that eigenvalues for X , with $Y = 0$, will come in $\frac{k}{2}$ plus and minus pairs. We can now

consider tuning the g_r coefficients in the superpotential (4.21) such that there is only one pair of plus and minus roots

$$X = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix},$$

where a depends on the g_r s. The vacuum expectation value of X breaks the gauge group into two parts $SU(\frac{N_c}{2}) \times SU(\frac{N_c}{2}) \times U(1)$. The dual gauge group, because traces of products of X and Y map to the same traces of products of \bar{X} and \bar{Y} , breaks to $SU(\frac{3kN_f - N_c}{2}) \times SU(\frac{3kN_f - N_c}{2}) \times U(1)$. The field X will decompose into an adjoint, we will call, X_1 charged under the first gauge group, an adjoint X_2 charged under the second gauge group, and some fields that are eaten by the Higgs mechanism. The field Y , however, decomposes into adjoints Y_1 and Y_2 and fields F and \tilde{F} that are not eaten and are in the $(\frac{N_c}{2}, \frac{N_c}{2})$ representation of the product gauge group. Under this decomposition, the superpotential becomes

$$\begin{aligned} W = & s_1(2a)^{\frac{k}{2}} \text{Tr } X_1^{\frac{k}{2}+1} + s_1(-2a)^{\frac{k}{2}} \text{Tr } X_2^{\frac{k}{2}+1} + s_2 a \text{Tr } Y_1^2 - s_2 a \text{Tr } Y_2^2 \\ & + s_2 \text{Tr } X_1 Y_1^2 + s_2 \text{Tr } X_2 Y_2^2 + s_2 \text{Tr } X_1 F \tilde{F} + s_2 \text{Tr } X_2 F \tilde{F} + \eta_3 \text{Tr } Y_1 F \tilde{F} \\ & + \eta_3 \text{Tr } Y_2 \tilde{F} F + \frac{\eta_3 \text{Tr } Y_1^3}{3} + \frac{\eta_3 \text{Tr } Y_2^3}{3} + \eta_4 \text{Tr } (F \tilde{F})^2 + \frac{\eta_4 \text{Tr } Y_1^4}{4} + \frac{\eta_4 \text{Tr } Y_2^4}{4} \\ & + \eta_4 \text{Tr } Y_1^2 F \tilde{F} + \eta_4 \text{Tr } Y_2^2 \tilde{F} F + \dots \end{aligned} \quad (4.23)$$

We see that the fields F do not get a mass. We can integrate out the massive fields Y_1 and Y_2 , and we are left with

$$\begin{aligned} W = & s_1(2a)^{\frac{k}{2}} \text{Tr } X_1^{\frac{k}{2}+1} + s_1(-2a)^{\frac{k}{2}} \text{Tr } X_2^{\frac{k}{2}+1} \\ & + s_2 \text{Tr } X_1 F \tilde{F} + s_2 \text{Tr } X_2 F \tilde{F} + \eta_4 \text{Tr } (F \tilde{F})^2. \end{aligned} \quad (4.24)$$

Thus, by deforming the superpotential and flowing to a lower energy theory, we have been led to a prediction for a new duality. In fact, it can be shown that the anomaly matching conditions still holds for the theory with superpotential (4.24). As in the high energy theory, the chiral ring is not truncated by the classical equations of motion in the way necessary for the duality. The necessary additional constraints, in the form of the Lagrange multipliers such as η_4 , were implemented in the high energy theory. The difference, noted earlier, between the k even and k odd cases has followed us in the renormalization group flow as might be expected.

We can now go back to superpotential (4.21) and ask what happens more generally when there are $\frac{k}{2}$ distinct plus and minus pairs of roots. The theory will break apart into

$\frac{k}{2}$ decoupled copies of the above described duality. The superpotential in one of these $\frac{k}{2}$ vacua is

$$W = s_1(2a) \prod_{i=1}^{\frac{k}{2}-1} (a - b_i)(a + b_i) \text{Tr } X_1^2 + s_1(-2a) \prod_{i=1}^{\frac{k}{2}-1} (a - b_i)(a + b_i) \text{Tr } X_2^2 \\ + s_2 \text{Tr } X_1 F \tilde{F} + s_2 \text{Tr } X_2 F \tilde{F} + \eta_4 \text{Tr } (F \tilde{F})^2. \quad (4.25)$$

Where a and b_i depend on the g_r s in (4.21). The electric gauge group breaks from $SU(N_c)$ down to $\prod_{i=1}^{\frac{k}{2}} SU(n_i) \times SU(n_i)$ while the magnetic gauge group break from $SU(3kN_f - N_c)$ down to $\prod_{i=1}^{\frac{k}{2}} SU(3N_f - n_i) \times SU(3N_f - n_i)$. Now, we can integrate out the massive fields X_1 and X_2 , and we are left with

$$W_L = \eta_4 \text{Tr } (F \tilde{F})^2 \quad (4.26)$$

which is just a superpotential of the Intriligator-Leigh-Strassler duality mentioned in the previous subsection. We can now add a mass term to the superpotential,

$$W_L = \eta_4 \text{Tr } (F \tilde{F})^2 + m \text{Tr } F \tilde{F}, \quad (4.27)$$

which breaks the theory down to $SU(p_0) \times U(p) \times SU(p_0)$ where there are N_f flavors charged under the $SU(p_0)$ s and $2N_f$ flavors charged under the diagonal subgroup $U(p)$. The dual theory breaks to $SU(N_f - p_0) \times U(2N_f - p) \times SU(N_f - p_0)$ [24]. There is an adjoint field charged under the diagonal subgroup $U(p)$ which must receive a mass if the duality is to be that of SQCD [10]. We see that the mass comes from the Lagrange multiplier term. Thus, the duality of the k evens is consistent if we impose additional constraints, not implied by the classical equations of motion.

5. A New Duality

This section can be read independently of the other sections. The deformations of the even k superpotential in the previous section led us to a prediction for a new duality. This duality can be generalized to one in which the electric theory has an $SU(N_c) \times SU(N'_c)$ gauge group with N_f fundamentals Q and N_f anti-fundamentals \tilde{Q} charged under only $SU(N_c)$, N'_f fundamentals Q' and N'_f anti-fundamentals \tilde{Q}' charged only under $SU(N'_c)$, a field F which is in an (N_c, N'_c) representation, a field \tilde{F} which is in an (\bar{N}_c, \bar{N}'_c) representation, a field X_1 which is in the adjoint representation of $SU(N_c)$, and a field X_2 which is in the adjoint representation of $SU(N'_c)$.

	$Q; \tilde{Q}$	$Q'; \tilde{Q}'$	$F; \tilde{F}$	X_1	X_2
$SU(N_c)$	$\mathbf{N}_c; \overline{\mathbf{N}}_c$	$\mathbf{1}; \mathbf{1}$	$\mathbf{N}_c; \overline{\mathbf{N}}_c$	adj	1
$SU(N'_c)$	$\mathbf{1}; \mathbf{1}$	$\mathbf{N}'_c; \overline{\mathbf{N}}'_c$	$\mathbf{N}'_c; \overline{\mathbf{N}}'_c$	1	adj
$SU(N_f)_L$	$\mathbf{N}_f; \mathbf{1}$	$\mathbf{1}; \mathbf{1}$	$\mathbf{1}; \mathbf{1}$	1	1
$SU(N'_f)_L$	$\mathbf{1}; \mathbf{1}$	$\mathbf{N}'_f; \mathbf{1}$	$\mathbf{1}; \mathbf{1}$	1	1
$SU(N_f)_R$	$\mathbf{1}; \mathbf{N}_f$	$\mathbf{1}; \mathbf{1}$	$\mathbf{1}; \mathbf{1}$	1	1
$SU(N'_f)_R$	$\mathbf{1}; \mathbf{1}$	$\mathbf{1}; \mathbf{N}'_f$	$\mathbf{1}; \mathbf{1}$	1	1
$U(1)_B$	$\pm \frac{1}{N_c}$	$\mp \frac{1}{N'_c}$	$\mp \left(\frac{1}{N_c} + \frac{1}{N'_c} \right)$	0	0
$U(1)_{B'}$	$\pm \frac{1}{N_c}$	$\pm \frac{1}{N'_c}$	$\mp \left(\frac{1}{N_c} - \frac{1}{N'_c} \right)$	0	0
$U(1)_F$	0	0	± 1	0	0
$U(1)_R$	$1 + \frac{(N'_c - 2N_c)}{N_f(k+1)}$	$1 + \frac{(N_c - 2N'_c)}{N'_f(k+1)}$	$\frac{k}{k+1}$	$\frac{2}{k+1}$	$\frac{2}{k+1}$

The superpotential is

$$\begin{aligned}
W = & \frac{s_1}{k+1} \text{Tr } X_1^{k+1} + \frac{s_1(-1)^{k+1}}{k+1} \text{Tr } X_2^{k+1} \\
& + s_2 \text{Tr } X_1 F \tilde{F} + s_2 \text{Tr } X_2 F \tilde{F} + \lambda_1 \text{Tr } X_1 + \lambda_2 \text{Tr } X_2
\end{aligned} \tag{5.1}$$

where the λ s are there to enforce the tracelessness of the adjoints. The equations of motion are now

$$\begin{aligned}
s_1 X_1^k + s_2 F \tilde{F} + \lambda_1 &= 0 \\
s_1 (-1)^{k+1} X_2^k + s_2 F \tilde{F} + \lambda_2 &= 0 \\
X_1 F + X_2 F &= 0 \\
X_1 \tilde{F} + X_2 \tilde{F} &= 0.
\end{aligned} \tag{5.2}$$

The equations of motion truncate the chiral ring in a manner similar to that discussed in section 2 for the k odd two adjoint case. Dropping the λ s and multiplying the first equation of motion (5.2) by \tilde{F} , we find

$$s_1 X_1^k \tilde{F} = -s_2 \tilde{F} F \tilde{F}. \tag{5.3}$$

The second equation effectively turns X_1 s into X_2 s leaving a minus sign each time. We then have

$$s_1 (-1)^k X_2^k \tilde{F} = -s_2 \tilde{F} F \tilde{F}; \tag{5.4}$$

comparing this with the last equation of motion in (5.2) we see that $\tilde{F}F\tilde{F} = 0$. By a similar calculation one can show that $F\tilde{F}F = 0$.

Owing to the superpotential, the mesons in the theory are $M_{l,1} = QX_1^{l-1}\tilde{Q}$, $P_{l,1} = Q'X_2^{l-1}\tilde{Q}'$, $M_{l,2} = QX_1^{l-1}F\tilde{Q}'$, $P_{l,2} = Q'X_2^{l-1}\tilde{F}\tilde{Q}$, $M_{l,3} = QF\tilde{F}X_1^{l-1}\tilde{Q}$, $P_{l,3} = Q'\tilde{F}FX_2^{l-1}\tilde{Q}'$, where $j = 1 \cdots k$

We can also form baryons by introducing dressed quarks

$$\begin{aligned}
Q_{(l,1)} &= X_1^{l-1}Q \\
Q_{(l,2)} &= X_1^{l-1}\tilde{F}Q' \\
Q_{(l,3)} &= X_1^{l-1}\tilde{F}FQ \\
Q'_{(l,1)} &= X_2^{l-1}Q' \\
Q'_{(l,2)} &= X_2^{l-1}FQ \\
Q'_{(l,3)} &= X_2^{l-1}F\tilde{F}Q'; \quad l = 1, \cdots k.
\end{aligned} \tag{5.5}$$

and then contracting the gauge indices on an $SU(N_c)$ epsilon tensor,

$$B^{(n_{1,1}, n_{2,1}, \cdots, n_{k,3})} = Q_{(1,1)}^{n_{1,1}} \cdots Q_{(k,3)}^{n_{k,3}}; \quad \sum_{l=1}^k \sum_{j=1}^3 n_{l,j} = N_c, \tag{5.6}$$

or on an $SU(N'_c)$ epsilon tensor to form

$$B'^{(n'_{1,1}, n'_{2,1}, \cdots, n'_{k,3})} = Q'_{(1,1)}^{n'_{1,1}} \cdots Q'_{(k,3)}^{n'_{k,3}}; \quad \sum_{l=1}^k \sum_{j=1}^3 n'_{l,j} = N'_c. \tag{5.7}$$

The total number of baryons is

$$\begin{aligned}
\sum_{\{n_{l,j}\}} \binom{N_f}{n_{1,1}} \cdots \binom{N'_f}{n_{k,2}} \cdots \binom{N_f}{n_{k,3}} + \sum_{\{n'_{l,j}\}} \binom{N'_f}{n'_{1,1}} \cdots \binom{N_f}{n'_{k,2}} \cdots \binom{N'_f}{n'_{k,3}} \\
= \binom{2kN_f + kN'_f}{N_c} + \binom{2kN'_f + kN_f}{N'_c}.
\end{aligned} \tag{5.8}$$

There are also anti-baryons and traces of the form $\text{Tr } X_1^r$ and $\text{Tr } X_2^r$.

5.1. duality

The dual theory has a gauge group $SU(2kN'_f + kN_f - N'_c) \times SU(2kN_f + kN'_f - N_c)$ with matter content :

	$q; \tilde{q}$	$q'; \tilde{q}'$	$\bar{F}; \tilde{\bar{F}}$	\bar{X}_1	\bar{X}_2
$SU(\tilde{N}_c)$	$\tilde{\mathbf{N}}_c; \overline{\tilde{\mathbf{N}}_c}$	$\mathbf{1}; \mathbf{1}$	$\tilde{\mathbf{N}}_c; \overline{\tilde{\mathbf{N}}_c}$	adj	1
$SU(\tilde{N}'_c)$	$\mathbf{1}; \mathbf{1}$	$\tilde{\mathbf{N}}'_c; \overline{\tilde{\mathbf{N}}'_c}$	$\tilde{\mathbf{N}}'_c; \overline{\tilde{\mathbf{N}}'_c}$	1	adj
$SU(N_f)_L$	$\mathbf{1}; \mathbf{1}$	$\overline{\mathbf{N}}_f; \mathbf{1}$	$\mathbf{1}; \mathbf{1}$	1	1
$SU(N_f)_R$	$\mathbf{1}; \mathbf{1}$	$\mathbf{1}; \overline{\mathbf{N}}_f$	$\mathbf{1}; \mathbf{1}$	1	1
$SU(N'_f)_L$	$\overline{\mathbf{N}}'_f; \mathbf{1}$	$\mathbf{1}; \mathbf{1}$	$\mathbf{1}; \mathbf{1}$	1	1
$SU(N'_f)_R$	$\mathbf{1}; \overline{\mathbf{N}}'_f$	$\mathbf{1}; \mathbf{1}$	$\mathbf{1}; \mathbf{1}$	1	1
$U(1)_B$	$\mp \frac{1}{\tilde{N}_c}$	$\pm \frac{1}{\tilde{N}'_c}$	$\pm \left(\frac{1}{\tilde{N}_c} + \frac{1}{\tilde{N}'_c} \right)$	0	0
$U(1)_{B'}$	$\pm \frac{1}{\tilde{N}_c}$	$\pm \frac{1}{\tilde{N}'_c}$	$\mp \left(\frac{1}{\tilde{N}_c} - \frac{1}{\tilde{N}'_c} \right)$	0	0
$U(1)_F$	$\mp \frac{N_f}{\tilde{N}_c}$	$\pm \frac{N'_f}{\tilde{N}'_c}$	$\mp \left(1 - \frac{N_f}{\tilde{N}_c} - \frac{N'_f}{\tilde{N}'_c} \right)$	0	0
$U(1)_R$	$1 + \frac{(\tilde{N}'_c - 2\tilde{N}_c)}{\tilde{N}'_f(k+1)}$	$1 + \frac{(\tilde{N}_c - 2\tilde{N}'_c)}{\tilde{N}_f(k+1)}$	$\frac{k}{k+1}$	$\frac{2}{k+1}$	$\frac{2}{k+1}$

and a dual superpotential of the form

$$\begin{aligned}
W = & \frac{s_1}{k+1} \text{Tr } \bar{X}_1^{k+1} + \frac{s_1(-1)^{k+1}}{k+1} \text{Tr } \bar{X}_2^{k+1} + s_2 \text{Tr } \bar{X}_1 \bar{F} \tilde{\bar{F}} + s_2 \text{Tr } \bar{X}_2 \tilde{\bar{F}} \tilde{\bar{F}} + \lambda_1 \text{Tr } \bar{X}_1 + \lambda_2 \text{Tr } \bar{X}_2 \\
& + \frac{s_1 s_2}{\mu^4} \sum_{l=1}^k M_{l,1} \tilde{q}' \bar{X}_2^{k-l} \bar{F} \tilde{\bar{F}} q' + P_{l,1} \tilde{q} \bar{X}_1^{k-l} \tilde{\bar{F}} \bar{F} q + P_{l,2} \tilde{q}' \bar{X}_2^{k-l} \bar{F} q \\
& + M_{l,2} \tilde{q} \bar{X}_1^{k-l} \tilde{\bar{F}} q' + M_{l,3} \tilde{q}' \bar{X}_2^{k-l} q' + P_{l,3} \tilde{q} \bar{X}_1^{k-l} q
\end{aligned} \tag{5.9}$$

The mesons are mapped to the singlets where as the baryon-like objects are mapped to other baryon-like objects in the dual theory: The mapping is

$$\begin{aligned}
B_{\text{el}}^{(n_{1,1}, n_{2,1}, \dots, n_{k,3})} & \leftrightarrow B'_{\text{mag}}^{(m'_{1,1}, m'_{2,1}, \dots, m'_{k,3})}; \quad m'_{l,1} = N_f - n_{k+1-l,3}; \\
& m'_{l,2} = N'_f - n_{k+1-l,2}; \\
& m'_{l,3} = N_f - n_{k+1-l,1}; \\
& l = 1, 2, \dots, k
\end{aligned} \tag{5.10}$$

$$\begin{aligned}
B'_{\text{el}}^{(n'_{1,1}, n'_{2,1}, \dots, n'_{k,3})} & \leftrightarrow B_{\text{mag}}^{(m_{1,1}, m_{2,1}, \dots, m_{k,3})}; \quad m_{l,1} = N'_f - n'_{k+1-l,3}; \\
& m_{l,2} = N_f - n'_{k+1-l,2}; \\
& m_{l,3} = N'_f - n'_{k+1-l,1}; \\
& l = 1, 2, \dots, k
\end{aligned} \tag{5.11}$$

The fact that this mapping is consistent with all global symmetries is another non-trivial test of the proposed duality. The t'Hooft anomaly matching conditions are satisfied for this duality.

We saw that the chiral ring was truncated in the way necessary for the duality for all values of k provided the factor $(-1)^{k+1}$ is in front of $\text{Tr } X_2^{k+1}$. It would be surprising if the duality depended critically on the numerical coefficients in the superpotential. We see again that the duality seems to suggest that the chiral ring is truncated by some other mechanism.

6. Conclusions.

One should make a distinction between the two cases: odd and even k . For odd k the two adjoint theory is has essentially the same features as the one adjoint theory. Classically, there were many flat directions that could be labeled by gauge invariant operators. A superpotential was added that lifted many of them and made the theory tractable. The classical equations of motion of the superpotential told us exactly which gauge invariants to keep and which ones we should set equal to zero. In the even k case, the classical equations of motion coming from the superpotential do not tell us which invariants to keep. We have seen that the duality holds if many of them are set to zero. Perhaps some of the gauge invariant mesons on the electric side could be mapped to some of the gauge invariant mesons on the magnetic side. In this way they would never appear as singlets in the dual theory. Such a mapping has not been found. It is possible that non-perturbative effects could truncate the chiral ring in a manner similar to that discussed in [18]. It would be interesting to see how this would work in the case considered here. One might speculate that a quantum truncation of the chiral ring might occur in the conformally invariant non-Abelian Coulomb phase where both theories are strongly coupled. The non-Abelian Coulomb phase is also the phase in which analogies with two dimensional conformal field theory would be most relevant.

We saw that it was the k even theories that led us to the new duality. However, in section 5, this duality was shown to be more general than than was evident when embedded in the k even theories, and in some instances the chiral ring can be shown to truncate classically.

Hopefully, this example will help us understand more about duality in supersymmetric gauge theories.

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7. Appendix.

Here we present a D-flat solution to (4.4) for the $k = 3$ theory. It is

$$X = \begin{pmatrix} 0 & \sqrt{17} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . \\ 0 & 0 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & . \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{3\sqrt{7}}{2} & 0 & 0 & . \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & . \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & . \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & . \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{17} & . \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . \\ . & . & . & . & . & . & . & . & . & . \end{pmatrix}$$

$$Y = \begin{pmatrix} 0 & 0 & 0 & \frac{\sqrt{17}}{2} & 0 & 0 & 0 & 0 & 0 & . \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & . \\ 0 & 0 & 0 & 0 & 0 & \frac{-1}{2} & 0 & 0 & 0 & . \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{2} & 0 & 0 & . \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & . \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{17}}{2} & . \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . \\ . & . & . & . & . & . & . & . & . & . \end{pmatrix}$$

with $\tilde{q}_\alpha^{N_f} = \frac{\sqrt{85}}{2}\delta_{\alpha,1}$ and $q_{N_f}^\alpha = \frac{\sqrt{85}}{2}\delta^{\alpha,9}$.

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